ON SUBGROUPS OF SATURATED OR TOTALLY BOUNDED PARATOPOLOGICAL GROUPS

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Abstract. A paratopological group G is *saturated* if the inverse U^{-1} of each non-empty set $U \subset G$ has non-empty interior. It is shown that a [first-countable] paratopological group H is a closed subgroup of a saturated (totally bounded) [abelian] paratopological group if and only if H admits a continuous bijective homomorphism onto a (totally bounded) [abelian] topological group G [such that for each neighborhood $U \subset H$ of the unit e there is a closed subset $F \subset G$ with $e \in h^{-1}(F) \subset U$]. As an application we construct a paratopological group whose character exceeds its π-weight as well as the character of its group reflexion. Also we present several examples of (para)topological groups which are subgroups of totally bounded paratopological groups but fail to be subgroups of *regular* totally bounded paratopological groups.

In this paper we continue investigations of paratopological groups, started by the authors in [Ra$_1$], [Ra$_2$], [Ra$_4$], [BR$_1$]–[BR$_3$]. By a *paratopological group* we understand a pair (G, τ) consisting of a group G and a topology τ on G making the group operation $\cdot : G \times G \to G$ of G continuous (such a topology τ will be called a *semigroup topology* on G). If, in addition, the operation $(\cdot)^{-1} : G \to G$ of taking the inverse is continuous with respect to the topology τ, then (G, τ) is a *topological group*. All topological spaces considered in this paper are supposed to be Hausdorff if the opposite is not stated.

The absence of the continuity of the inverse in paratopological groups results in appearing various pathologies impossible in the category of topological groups, which makes the theory of paratopological groups quite interesting and unpredictable. In [Gu] I. Guran has introduced a relatively narrow class of so-called saturated paratopological groups which behave much like usual topological groups. Following I.Guran we define a paratopological group G to be *saturated* if the inverse U^{-1} of any nonempty open subset U of G has non-empty interior in G. A standard example of a saturated paratopological group with discontinuous inverse is the *Sorgenfrey line*, that is the real line endowed with the Sorgenfrey topology generated by the base consisting of half-intervals $[a, b)$, $a < b$. Important examples of saturated paratopological groups are *totally bounded* groups, that is paratopological groups G such that for any neighborhood $U \subset G$ of the origin in G there is a finite subset $F \subset G$ with $G = FU = UF$, see [Ra$_3$, Proposition 3.1].

Observing that each subgroup of the Sorgenfrey line is saturated, I.Guran asked if the same is true for any saturated paratopological group. This question can be also posed in another way: which paratopological groups embed into saturated ones? A similar question concerning embedding into totally bounded semi- or paratopological groups appeared in [AH].

In this paper we shall show that the necessary and sufficient condition for a paratopological group G to embed into a saturated (totally bounded) paratopological group is

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the existence of a bijective continuous group homomorphism $h : G \to H$ onto a topological (totally bounded) group H. The latter property of G will be referred to as the $\flat$-separateness (resp. Bohr separateness).

$\flat$-Separated paratopological groups can be equivalently defined with help of the group reflexion of a paratopological group. Given a paratopological group G let $\tau^\flat$ be the strongest group topology on G, weaker than the topology of G. The topological group $G^\flat = (G, \tau^\flat)$, called *the group reflexion* of G, has the following characteristic property: the identity map $i : G \to G^\flat$ is continuous and for every continuous group homomorphism $h : G \to H$ from G into a topological group H the homomorphism $h \circ i^{-1} : G^\flat \to H$ is continuous. According to [BR$_1$], a neighborhood base of the unit of the group reflexion $G^\flat$ of a *saturated* (more generally, 2-oscillating) paratopological group G consists of the sets UU^{-1} where U runs over neighborhoods of the unit in G. For instance, the group reflexion of the Sorgenfrey line is the usual real line.

There is also a dual notion of a group co-reflexion. Given a paratopological group G let $\tau_\sharp$ be the weakest group topology on G, stronger than the topology of G. The topological group $G^\sharp = (G, \tau_\sharp)$ is called *the group co-reflexion* of G. According to [Ra$_1$], a neighborhood base of the unit of the group co-reflexion $G^\sharp$ of a paratopological group G consists of the sets $U \cap U^{-1}$ where U runs over neighborhoods of the unit in G. A paratopological group is called $\sharp$-*discrete* provided its group co-reflection is discrete. For instance, the Sorgenfrey line is $\sharp$-discrete.

A subset A of a paratopological group G will be called $\flat$-*closed* if A is closed in the topology $\tau^\flat$. A paratopological group G is called $\flat$-*separated* provided its group reflexion $G^\flat$ is Hausdorff; G is called $\flat$-*regular* if each neighborhood U of the unit e of G contains a $\flat$-closed neighborhood of e. It is easy to see that each $\flat$-regular paratopological group is regular and $\flat$-separated, see [BR$_1$]. For saturated groups the converse is also true: each regular saturated paratopological group is $\flat$-regular, see [BR$_1$, Theorem 3].

The notions of a $\flat$-separated (resp. $\flat$-regular) paratopological group is a partial case of a more general notion of a $\mathcal{G}$-separated (resp. $\mathcal{G}$-regular) paratopological group, where $\mathcal{G}$ is a class of topological groups. Namely, we call a paratopological group G to be

- $\mathcal{G}$-*separated* if G admits a continuous bijective homomorphism $h : G \to H$ onto a topological group $H \in \mathcal{G}$;
- $\mathcal{G}$-*regular* if G admits a regular continuous homomorphism $h : G \to H$ onto a topological group $H \in \mathcal{G}$.

A continuous map $h : X \to Y$ between topological spaces is defined to be *regular* if for each point $x \in X$ and a neighborhood U of x in X there is a closed subset $F \subset Y$ such that $h^{-1}(F)$ is a closed neighborhood of x with $h^{-1}(F) \subset U$.

As we shall see later, any injective continuous map from a k_ω-space is regular. We remind that a topological space X is defined to be a k_ω-*space* if there is a countable cover $\mathcal{K}$ of X by compact subsets of X, determining the topology of X in the sense that a subset U of X is open in X if and only if the intersection $U \cap K$ is open in K for any compact set $K \in \mathcal{K}$. According to [FT], each Hausdorff k_ω-space is normal. Under a *(para)topological k_ω-group* we shall understand a (para)topological group whose underlying topological space is k_ω. Many examples of k_ω-spaces appear in topological algebra as free objects in various categories, see [Cho]. In particular, the free (abelian) topological group over a compact Hausdorff space is a topological k_ω-group, see [Gra].

Proposition 1. *Any injective continuous map $f : X \to Y$ from a Hausdorff k_ω-space X into a Hausdorff space Y is regular.*

Proof. Fix any point $x \in X$ and an open neighborhood $U \subset X$ of x. Let $\{K_n\}$ be an increasing sequence of compacta determining the topology of the space X. Without loss of generality, we may assume that $K_0 = \{x\}$. Let $V_0 = K_0$ and $W_0 = Y$.

By induction, for every $n \geq 1$ we shall find an open neighborhood V_n of x in K_n and an open neighborhood W_n of $f(\overline{V}_n)$ in Y such that

1) $f^{-1}(\overline{W}_n) \cap K_n \subset U$;
2) $\overline{V}_n \subset U \cap \bigcap_{i \leq n} f^{-1}(W_i)$;
3) $\overline{V}_n \subset V_{n+1}$.

Assume that for some n the sets V_i, W_i, $i < n$, have been constructed. It follows from (2) that $f(\overline{V}_{n-1})$ and $f(K_n \setminus U)$ are disjoint compact sets in the Hausdorff space Y. Consequently, the compact set $f(\overline{V}_{n-1})$ has an open neighborhood $W_n \subset Y$ whose closure $\overline{W}_n$ misses the compact set $f(K_n \setminus U)$. Such a choice of W_n and the condition (2) imply that $\overline{V}_{n-1} \subset U \cap \bigcap_{i \leq n} f^{-1}(W_i)$. Using the normality of the compact space K_n find an open neighborhood V_n of $\overline{V}_{n-1}$ in K_n such that $\overline{V}_n \subset U \cap \bigcap_{i \leq n} f^{-1}(W_i)$, which finishes the inductive step.

It is easy to see that $V = \bigcup_{n \in \omega} V_n$ is an open neighborhood of x such that $f^{-1}(\overline{f(V)}) \subset f^{-1}(\bigcap_{i \in \omega} \overline{W}_i) \subset U$, which just yields the regularity of the map f. □

For paratopological k_ω-groups this proposition yields the equivalence between the $\mathcal{G}$-regularity and $\mathcal{G}$-separateness.

Corollary 1. *Let $\mathcal{G}$ be a class of topological groups. A paratopological k_ω-group is $\mathcal{G}$-separated if and only if it is $\mathcal{G}$-regular.* □

A class $\mathcal{G}$ of topological groups will be called

- *closed-hereditary* if each closed subgroup of a group $G \in \mathcal{G}$ belongs to the class $\mathcal{G}$;
- *H-stable*, where H is a topological group, if $G \times H \in \mathcal{G}$ for any topological group $G \in \mathcal{G}$.

For a topological space X by $\chi(X)$ we denote its *character*, equal to the smallest cardinal κ such that each point $x \in X$ has a neighborhood base of size $\leq \kappa$.

Now we are able to give a characterization of subgroups of saturated paratopological groups (possessing certain additional properties).

Theorem 1. *Suppose T is a saturated $\sharp$-discrete nondiscrete paratopological group and $\mathcal{G}$ is a closed-hereditary $T^\flat$-stable class of topological groups. A paratopological group H is $\mathcal{G}$-separated if and only if H is a $\flat$-closed subgroup of a saturated paratopological group G with $G^\flat \in \mathcal{G}$, $\chi(G) = \max\{\chi(H), \chi(T)\}$, and $|G/H| = |T|$.*

A similar characterization holds for $\mathcal{G}$-regular paratopological groups. We remind that a paratopological group G is called a *paratopological SIN-group* if any neighborhood U of the unit e in G contains a neighborhood $W \subset G$ of e such that $gWg^{-1} \subset U$ for all $g \in G$. It is easy to check that every paratopological SIN-group has a base at the unit consisting of invariant open neighborhoods, see [Ra$_3$, Ch.4] (as expected, a subset A of a group G is called *invariant* if $xAx^{-1} = A$ for all $x \in G$).

Finally we define a notion of a Sorgenfrey paratopological group which crystallizes some important properties of the Sorgenfrey topology on the real line. A paratopological group G is defined to be *Sorgenfrey* if G is non-discrete, saturated and contains a neighborhood

U of the unit e such that for any neighborhood $V \subset G$ of e there is a neighborhood $W \subset G$ of e such that $x, y \in V$ for any elements $x, y \in U$ with $xy \in W$. Observe that each Sorgenfrey paratopological group is $\sharp$-discrete.

Theorem 2. *Suppose T is a first countable saturated regular Sorgenfrey paratopological SIN-group and $\mathcal{G}$ is a closed-hereditary $T^\flat$-stable class of first countable topological SIN-groups. A paratopological group H is $\mathcal{G}$-regular if and only if H is a $\flat$-closed subgroup of a saturated $\flat$-regular paratopological group G with $G^\flat \in \mathcal{G}$, $\chi(G) = \chi(H)$, and $|G/H| = |T|$.*

Theorem 2 in combination with Corollary 1 yield

Corollary 2. *Suppose T is a first countable saturated regular Sorgenfrey paratopological SIN-group and $\mathcal{G}$ is a closed-hereditary $T^\flat$-stable class of first countable topological SIN-groups. A paratopological k_ω-group H is $\mathcal{G}$-separated if and only if H is a $\flat$-closed subgroup of a saturated $\flat$-regular paratopological group G with $G^\flat \in \mathcal{G}$, $\chi(G) = \chi(H)$, and $|G/H| = |T|$.* □

As we said, any (regular) saturated paratopological group is $\flat$-separated (and $\flat$-regular), see [BR$_1$]. Observe that a paratopological group G is $\flat$-separated if and only if G is TopGr-separated where TopGr stands for the class of all Hausdorff topological groups. These observations and Theorem 1 imply

Corollary 3. *A paratopological group H is $\flat$-separated if and only if H is a ($\flat$-closed) subgroup of a saturated paratopological group.* □

Unfortunately we do not know the answer to the obvious $\flat$-regular version of the above corollary.

Problem 1. *Is every $\flat$-regular paratopological group a subgroup of a regular saturated paratopological group?*

For first-countable paratopological SIN-groups the answer to this problem is affirmative.

Corollary 4. *A first-countable paratopological SIN-group H is $\flat$-regular if and only if H is a $\flat$-closed subgroup of a regular first-countable saturated paratopological SIN-group G with $|G/H| = \aleph_0$.*

Proof. Taking into account that H is a first-countable paratopological SIN-group and applying [BR$_1$, Proposition 3], we conclude that $H^\flat$ is a first-countable topological SIN-group. Let T be the quotient group $\mathbb{Q}/\mathbb{Z}$ of the group of rational numbers, endowed with the Sorgenfrey topology generated by the base consisting of half-intervals. Observe that T is a $\flat$-regular Sorgenfrey abelian paratopological group and the class of first countable topological SIN-groups is $T^\flat$ stable, closed-hereditary and contains $H^\flat$. Now to finish the proof apply Theorem 2. □

Next, we apply Theorems 1 and 2 to the class TBG (resp. FCTBG) of (first countable) totally bounded topological groups. We remind that a paratopological group G is *totally bounded* if for any neighborhood U of the unit in G there is a finite subset $F \subset G$ with $UF = FU = G$. It is known that each totally bounded paratopological group is saturated and a saturated paratopological group G is totally bounded if and only if its group reflexion $G^\flat$ is totally bounded, see [Ra$_2$], [BR$_1$]. An example of a $\flat$-regular totally bounded Sorgenfrey paratopological group is the circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ endowed with the Sorgenfrey topology generated by the base consisting of half-intervals $\{z \in \mathbb{T} : a \le \mathrm{Arg}(z) < b\}$ where $0 \le a < b \le 2\pi$.

We define a paratopological group G to be *Bohr separated* (resp. *Bohr regular*, *fcBohr regular*) if it is TBG-separated (resp. TBG-regular, FCTBG-regular). In this terminology Theorem 1 implies

Corollary 5. *A paratopological group H is Bohr separated if and only if H is a ($\flat$-closed) subgroup of a totally bounded paratopological group.* □

It is interesting to compare Corollary 5 with another characterizing theorem supplying us with many pathological examples of pseudocompact paratopological groups. We remind that a topological space X is *pseudocompact* if each locally finite family of non-empty open subsets of X is finite. It should be mentioned that a Tychonoff space X is pseudocompact if and only if each continuous real-valued function on X is bounded.

Theorem 3. *An abelian paratopological group H is Bohr separated if and only if H is a subgroup of a pseudocompact abelian paratopological group G with $\chi(G) = \chi(H)$.*

The following characterization of fcBohr regular paratopological groups follows from Theorem 2 applied to the class $\mathcal{G} =$ FCTBG and the quotient group $T = \mathbb{Q}/\mathbb{Z}$ endowed with the standard Sorgenfrey topology.

Corollary 6. *A paratopological group H is fcBohr regular if and only if H is a $\flat$-closed subgroup of a regular totally bounded paratopological group G with $\aleph_0 = \chi(G^\flat) \le \chi(G) = \chi(H)$ and $|G/H| \le \aleph_0$.*

In some cases the fcBohr (= FCTBG) regularity is equivalent to the Bohr (= TBG) regularity. We recall that a topological space X has *countable pseudocharacter* if each one point subset of X is a G_δ-subset.

Proposition 2. *A Bohr regular paratopological group H is fcBohr regular provided one of the following conditions is satisfied:*

(1) *H is a k_ω-space with countable pseudocharacter;*
(2) *H is first countable and Lindelöf.*

Proof. Using the Bohr regularity of H, find a regular bijective continuous homomorphism $h : H \to K$ onto a totally bounded topological group K. Denote by e_H and e_K the units of the groups H, K, respectively.

1. First assume that H is a k_ω-space with countable pseudo-character. In this case the set $H \setminus \{e_H\}$ is σ-compact as well as its image $h(H \setminus \{e_H\}) = K \setminus \{e_K\}$. It follows that the totally bounded group K has countable pseudo-character. Now it is standard to find a bijective continuous homomorphism $i : K \to G$ of K onto a first countable totally bounded topological group G, see [Tk, 4.5]. Since the composition $f \circ h : H \to G$ is bijective, the group H is FCTBG-separated and by Proposition 1 is fcBohr regular.

2. Next assume that H is first-countable and Lindelöf. Fix a sequence $(U_n)_{n\in\omega}$ of open subsets of H forming a neighborhood base at e_H. For every $n \in \omega$ fix a closed neighborhood $W_n \subset U_n$ whose image $h(W_n)$ is closed in K. Let us call an open subset $U \subset K$ *cylindrical* if $U = f^{-1}(V)$ for some continuous homomorphism $f : K \to G$ into a first countable compact topological group G and some open set $V \subset G$. It follows from the total boundedness of K that open cylindrical subsets form a base of the topology of the group K, see [Tk, 3.4].

Using the Lindelöf property of the set $f(H \setminus U_n)$, for every $n \in \omega$ find a countable cover $\mathcal{U}_n$ of $f(H \setminus U_n)$ by open cylindrical subsets such that $\cup\mathcal{U}_n \cap h(W_n) = \emptyset$. Then

$\mathcal{U} = \bigcup_{n\in\omega} \mathcal{U}_n$ is a countable collection of open cylindrical subsets and we can produce a continuous homomorphism $f : K \to G$ onto a first countable totally bounded topological group such that each set $U \in \mathcal{U}$ is the preimage $U = f^{-1}(V)$ of some open set $V \subset K$. To finish the proof it rests to observe that the composition $f \circ h : H \to G$ is a regular bijection of H onto a first countable totally bounded topological group. □

It can be shown that the character of any non-locally compact paratopological k_ω-group with countable pseudo-character is equal to the small cardinal $\mathfrak{d}$, well-known in the Modern Set Theory, see [JW], [Va]. By definition, $\mathfrak{d}$ is equal to the cofinality of $\mathbb{N}^\omega$ endowed with the natural partial order: $(x_i)_{i\in\omega} \le (y_i)_{i\in\omega}$ iff $x_i \le y_i$ for all i. More precisely, $\mathfrak{d}$ is equal to the smallest size of a subset $C \subset \mathbb{N}^\omega$ cofinal in the sense that for any $x \in \mathbb{N}^\omega$ there is $y \in C$ with $y \ge x$. It is easy to see that $\aleph_1 \le \mathfrak{d} \le \mathfrak{c}$. The Martin Axiom implies $\mathfrak{d} = \mathfrak{c}$. On the other hand, there are models of ZFC with $\mathfrak{d} < \mathfrak{c}$, see [Va].

We shall apply Corollary 6 to construct examples of paratopological groups whose character exceed their π-weight as well as the character of their group reflexions. We recall that *the π-weight* $\pi w(X)$ of a topological space X is the smallest size of a *π-base*, i.e., a collection $\mathcal{B}$ of open nonempty subsets of X such that each nonempty open subset U of X contains an element of the family $\mathcal{B}$. According to [Tk, 4.3] the π-weight of each topological group coincides with its weight.

Corollary 7. *For any uncountable cardinal $\kappa \le \mathfrak{d}$ there is a $\flat$-regular totally bounded countable abelian paratopological group G with $\aleph_0 = \chi(G^\flat) = \pi w(G) < \chi(G) = \kappa$.*

Proof. Take any non-metrizable countable abelian FCTBG-separated topological k_ω-group (H, τ) (for example, let H be a free abelian group over a convergent sequence, see [FT]). The group H, being FCTBG-separated, admits a bijective continuous homomorphism $h : H \to K$ onto a first countable totally bounded abelian topological group K. By Proposition 1 this homomorphism is regular.

According to [FT, 22)] or [Ban], the space H contains a copy of the Fréchet-Urysohn fan S_ω and thus has character $\chi(H) \ge \chi(S_\omega) \ge \mathfrak{d}$. Using this fact and the countability of H, by transfinite induction (over ordinals $< \kappa$) construct a group topology $\sigma \subset \tau$ on H such that $\chi(H, \sigma) = \kappa$ and the homomorphism $h : (H, \sigma) \to K$ is regular. This means that the group (H, σ) is fcBohr regular. Now we can apply Corollary 6 to embed the group (H, σ) into a totally bounded countable abelian paratopological group G such that $\aleph_0 = \chi(G^\flat) < \chi(G) = \chi(H, \sigma) = \kappa$. Since G is saturated and abelian, $\flat$-open subsets of G form a π-base which implies $\pi w(G) = \omega(G^\flat) = \aleph_0$. □

Remark 1. It is interesting to compare Corollary 7 with a result of [BRZ] asserting that there exists a $\flat$-regular countable paratopological group G with $\aleph_0 = \chi(G) < \chi(G^\flat) = \mathfrak{d}$. Such a paratopological group G cannot be saturated since $\chi(G^\flat) \le \chi(G)$ for any saturated (more generally, any 2-oscillating) paratopological group G, see [BR$_1$].

We finish our discussion with presenting examples of regular (para)topological groups which embed into totally bounded paratopological groups but fail to embed into *regular* totally bounded paratopological groups. For that it suffices to find a Bohr separated group which is not Bohr regular.

Let us remark that each locally convex linear topological space (or more generally each locally quasi-convex abelian topological group, see [A] or [Ba]) is Bohr regular. On the other hand, there exist (non-locally convex) linear metric spaces which fail to be Bohr separated or Bohr regular. The simplest example can be constructed as follows.

Consider the linear space $C[0,1]$ of continuous real-valued functions on the closed interval $[0,1]$ and endow it with the invariant metrics $d_{1/2}(f,g)=\int_0^1\sqrt{|f(t)-g(t)|}dt$, $p(f,g)=\sum_{n\in\omega}\min\{2^{-n},|f(t_n)-g(t_n)|\}$ and $\rho(f,g)=d_{1/2}(f,g)+p(f,g)$ where $\{t_n:n\in\omega\}$ is an enumeration of rational numbers of $[0,1]$. It is well-known that the linear metric space $(C[0,1],d_{1/2})$ admits no non-zero linear continuous functional and fails to be Bohr separated. The linear metric space $(C[0,1],\rho)$ is even more interesting. We remind that an abelian group G is *divisible* (resp. *torsion-free*) if for any $a\in G$ and natural n the equation $x^n=a$ has a solution (resp. has at most one solution) $x\in G$.

Proposition 3. *The linear metric space $L=(C[0,1],\rho)$ is Bohr separated but fails to be Bohr regular. Moreover, L is a $\flat$-closed subgroup of a totally bounded abelian torsion-free divisible group, but fails to be a subgroup of a* regular *totally bounded paratopological group.*

Proof. The Bohr separatedness of L follows from the continuity of the maps $\chi_n:L\to\mathbb{R}$, $\chi_n:f\mapsto f(t_n)$, for $n\in\omega$. Let us show that the group L fails to be Bohr regular.

For this we first prove that each linear continuous functional $\psi:(L,\rho)\to\mathbb{R}$ is continuous with respect to the "product" metric p. Consider the open convex subset $C=\psi^{-1}(-1,1)$ of L. By the continuity of ψ, there are $n\ge 1$ and $\varepsilon>0$ such that $x\in C$ for any $x\in L$ with $d_{1/2}(x,0)<\varepsilon$ and $|x(t_i)|<\varepsilon$ for all $i\le n$. Let $L_0=\{x\in L:x(t_i)=0$ for all $i\le n\}$ and observe that the convex set $C\cap L_0$ contains the open ε-ball with respect to the restriction of the metric $d_{1/2}$ on L_0. Now the standard argument (see [Ed, 4.16.3]) yields $C\cap L_0=L_0$ and $L_0\subset\bigcap_{k\ge1}\frac{1}{k}C=\psi^{-1}(0)$. Hence the functional ψ factors through the quotient space L/L_0 and is continuous with respect to the metric p (this follows from the continuity of the quotient homomorphism $L\to L/L_0$ with respect to p).

If $\chi:L\to\mathbb{T}$ is any character on L (that is a continuous group homomorphism into the circle $\mathbb{T}=\mathbb{R}/\mathbb{Z}$), then it is easy to find a continuous linear functional $\psi:L\to\mathbb{R}$ such that $\chi=\pi\circ\psi$, where $\pi:\mathbb{R}\to\mathbb{T}$ is the quotient homomorphism. As we have already shown, the functional ψ is continuous with respect to the metric p and so is the character χ.

Finally, we are able to prove that the group L fails to be Bohr regular. Assuming the converse we would find a continuous regular homomorphism $h:L\to H$ onto a totally bounded abelian topological group H. The group H, being abelian and totally bounded, is a subgroup of the product $\mathbb{T}^\kappa$ for some cardinal κ, see [Mo]. Then the above discussion yields that h is continuous with respect to the metric p. In this situation the regularity of h implies the regularity of the identity map $(L,\rho)\to(L,p)$. But this map certainly is not regular: for any 2^{-n}-ball $B=\{x\in L:\rho(x,0)<2^{-n}\}$ its closure in the metric p contains the linear subspace $\{x\in L:x(t_i)=0$ for all $i\le n\}$ and thus lies in no ball. Therefore the group L is Bohr separated but not Bohr regular.

Let $\mathcal{G}$ be the class of all totally bounded abelian divisible torsion-free topological groups. The group L, being Bohr separated, abelian, divisible, and torsion-free, is $\mathcal{G}$-separated. Pick any irrational number $\alpha\in\mathbb{T}=\mathbb{R}/\mathbb{Z}$ and consider the subgroup $T=\{q\alpha:q\in\mathbb{Q}\}$ of the circle $\mathbb{T}$ endowed with the Sorgenfrey topology. It is clear that T is a totally bounded $\sharp$-discrete paratopological group with $T^\flat\in\mathcal{G}$. By Theorem 1, L is a $\flat$-closed subgroup of a saturated paratopological groups G with $G^\flat\in\mathcal{G}$ which implies that G is totally bounded abelian, divisible and torsion-free.

On the other hand, L admits no embedding into a regular totally bounded paratopological group G. Indeed, assuming that $L\subset G$ is such an embedding, apply Theorem 3

of [BR$_1$] to conclude that the identity homomorphism $id : G \to G^\flat$ is regular and so is its restriction $id|L$, which would imply the Bohr regularity of L. □

There is also an alternative method of constructing Bohr separated but not Bohr regular paratopological groups, based on the concept of a Lawson paratopological group. Following [BR$_1$] we define a paratopological group G to be *Lawson* if it has a neighborhood base at the unit consisting of subsemigroups of G. According to [BR$_1$] there is a regular Lawson paratopological group failing to be $\flat$-separated. On the other hand, there are Lawson paratopological groups which are $\flat$-regular and Bohr separated but are not topological groups, see Example 2 [BR$_1$] or Example 1 below. We shall show that a $\flat$-regular paratopological group G is a topological group provided its group reflexion $G^\flat$ is topologically periodic. We remind that a paratopological group G is *topologically periodic* if for each $x \in G$ and a neighborhood $U \subset G$ of the unit there is a number $n \geq 1$ such that $x^n \in U$, see [BG]. It is easy to show that each totally bounded topological group is topologically periodic. For paratopological groups it is not true: according to Theorem 2 there is a $\flat$-regular totally bounded paratopological group G which contains the discrete group $\mathbb{Z}$ of integers and thus cannot be topologically periodic. The class of topologically periodic topological groups will be denoted by TPTG.

Proposition 4. *Each* TPTG*-regular Lawson paratopological group is a topological group.*

Proof. Let (G, τ) be a Lawson paratopological group and $\sigma \subset \tau$ be a topology turning G into a topologically periodic topological group such that (G, τ) has a base $\mathcal{B}$ at the unit consisting of subsemigroups, closed in the topology σ. We are going to show that an arbitrary element $U \in \mathcal{B}$ is in fact a subgroup of G. For this purpose suppose that there exists an element $x \in U^{-1} \backslash U$. Then $x^{-1} \in U$ and $x^m \in U$ for all $m < 0$ because U is a subsemigroup of G. Since the set U is closed in the topology σ, there exists a neighborhood $V \in \sigma$ of unit such that $xV \cap U = \emptyset$. By the topological periodicity of (G, σ), there exists a number $n < -1$ with $x^n \subset V$. Then $x^{n+1} \cap U = \emptyset$ which is a contradiction. □

Since each totally bounded topological group is topologically periodic this Proposition implies

Corollary 8. *Each Bohr regular Lawson paratopological group is a topological group.* □

On the other hand, *abelian* Lawson paratopological groups are Bohr separated.

Proposition 5. *Each abelian Lawson paratopological group is Bohr separated.*

Proof. Let G be such the group. Then $G^\flat$ has a neighborhood base $\mathcal{B}$ at the unit, consisting of subgroups. For every group $H \in \mathcal{B}$ the group G/H, being abelian and discrete, is Bohr separated [Mo]. Since the family $\{G \to G/H : H \in \mathcal{B}\}$ of quotient maps separates the points of the group G, the group G is Bohr separated too. □

Corollary 8 and Proposition 5 allow us to construct simple examples of Bohr separated Lawson paratopological groups which are not Bohr regular.

Example 1. *There is a countable $\flat$-regular saturated Lawson paratopological abelian group H which is Bohr separated but not Bohr regular. The group H has the following properties:*

(1) *H is a $\flat$-closed subgroup of a countable first-countable abelian totally bounded paratopological group;*

(2) *H is a $\flat$-closed subgroup of a first-countable abelian pseudocompact paratopological group;*

(3) *H fails to be a subgroup of a regular totally bounded (or pseudocompact) paratopological group.*

Proof. Consider the direct sum $\mathbb{Z}^\omega_0 = \{(x_i)_{i\in\omega} \in \mathbb{Z}^\omega : x_i = 0$ for all but finitely many indices $i\}$ of countably many copies of the group $\mathbb{Z}$ of integers. Endow the group $\mathbb{Z}^\omega_0$ with a shift invariant topology τ whose neighborhood base at the origin consists of the sets $U_n = \{0\} \cup \bigcup_{m\ge n} W_m$ where $W_m = \{(x_i)_{i\in\omega} \in \mathbb{Z}^\omega_0 : x_i = 0$ for all $i < m$ and $x_m > 0\}$ for $m \ge 0$. It is easy to see that $H = (\mathbb{Z}^\omega_0, \tau)$ is a $\flat$-regular countable first-countable saturated Lawson paratopological group which is not a topological group. By Proposition 5 and Corollary 8 the group H is Bohr separated but not Bohr regular.

By Theorem 1, H is a $\flat$-closed subgroup of a first-countable totally bounded countable paratopological group and by Theorem 3, H is a $\flat$-closed subgroup of a first-countable abelian pseudocompact paratopological group.

Assuming that H is a subgroup of a regular totally bounded or pseudocompact paratopological group G and applying Theorem 3 of [BR$_1$] and [RR] we would get that both G and H are Bohr regular which is impossible. □

In the proofs of our principal results we shall often exploit the following characterization of semigroup topologies on groups from [Ra$_1$, 1.1].

Lemma 1. *A family $\mathcal{B}$ of subsets containing a unit e of a group G is a neighborhood base at e of some semigroup topology τ on G if and only if $\mathcal{B}$ satisfies the following four Pontryagin conditions:*

1. $(\forall U, V \in \mathcal{B})(\exists W \in \mathcal{B}) : W \subset U \cap V$;
2. $(\forall U \in \mathcal{B})(\exists V \in \mathcal{B}) : V^2 \subset U$;
3. $(\forall U \in \mathcal{B})(\forall x \in U)(\exists V \in \mathcal{B}) : xV \subset U$;
4. $(\forall U \in \mathcal{B})(\forall x \in G)(\exists V \in \mathcal{B}) : x^{-1}Vx \subset U$.

The topology τ is Hausdorff if and only if

5. $\bigcap\{UU^{-1} : U \in \mathcal{B}\} = \{e\}$.

1. Proof of Theorem 1

The necessity is evident. We shall prove the sufficiency. Let (H, τ) be a $\mathcal{G}$-separated paratopological group, where $\mathcal{G}$ is $T^\flat$-stable class of topological groups. Since the group H is $\mathcal{G}$-separated, there exists a group topology σ on the group H such that $(H, \sigma) \in \mathcal{G}$. We shall define the topology on the product $G = H \times T$ as follows. Let $\mathcal{B}_\tau$, $\mathcal{B}_\sigma$ and $\mathcal{B}_T$ be open bases at the unit of the groups (H, τ), (H, σ) and T respectively. For arbitrary neighborhoods $U_\tau \in \mathcal{B}_\tau$, $U_\sigma \in \mathcal{B}_\sigma$ and $U_T \in \mathcal{B}_T$ with $U_\tau \subset U_\sigma$ put $[U_\tau, U_\sigma, U_T] = U_\tau \times \{e_T\} \cup U_\sigma \times (U_T \backslash \{e_T\})$, where e_H and e_T are the units of the groups H and T respectively. The family of all such $[U_\tau, U_\sigma, U_T]$ will be denoted by $\mathcal{B}$. Now we verify the Pontryagin conditions for the family $\mathcal{B}$.

The Condition 1 is trivial.

To check Condition 2 consider an arbitrary set $[U_\tau, U_\sigma, U_T] \in \mathcal{B}$. There exist neighborhoods $V_\tau \in \mathcal{B}_\tau$, $V_\sigma \in \mathcal{B}_\sigma$ such that $V_\tau^2 \subset U_\tau$, $V_\sigma^2 \subset U_\sigma$ and $V_\tau \subset V_\sigma$. Since the group $T^\#$ is discrete then there is a neighborhood $V_T \subset \mathcal{B}_T$ such that $(V_T \backslash \{e_T\})^2 \subset U_T \backslash \{e_T\}$. Then $[V_\tau, V_\sigma, V_T]^2 \subset [U_\tau, U_\sigma, U_T]$.

To verify Condition 3 consider an arbitrary point $x \in [U_\tau, U_\sigma, U_T] \in \mathcal{B}$. If $x = (x_H, e_T)$, where $x_H \in U_\tau$ then there exist neighborhoods $V_\tau \in \mathcal{B}_\tau$, $V_\sigma \in \mathcal{B}_\sigma$ such that $V_\tau \subset V_\sigma$,

$x_H V_\tau \subset U_\tau$ and $x_H V_\sigma \subset U_\sigma$. Then $x[V_\tau, V_\sigma, U_T] \subset [U_\tau, U_\sigma, U_T]$. If $x = (x_H, x_T)$, where $x_H \in U_\sigma$ and $x_T \in U_T \backslash \{e_T\}$ then there exist neighborhoods $V_\tau \in \mathcal{B}_\tau$, $V_\sigma \in \mathcal{B}_\sigma$ and $V_T \in \mathcal{B}_T$ such that $V_\tau \subset V_\sigma$, $x_H V_\sigma \subset U_\sigma$ and $x_T V_T \subset U_T \backslash \{e_T\}$. Then $x[V_\tau, V_\sigma, V_T] \subset [U_\tau, U_\sigma, U_T]$.

Condition 4. Let $x = (x_H, x_T) \subset H \times T$ be an arbitrary point. Then there are neighborhoods $V_\tau \in \mathcal{B}_\tau$, $V_\sigma \in \mathcal{B}_\sigma$ and $V_T \in \mathcal{B}_T$ such that $V_\tau \subset V_\sigma$, $x_H^{-1} V_\tau x_H \subset U_\tau$, $x_H^{-1} V_\sigma x_H \subset U_\sigma$ and $x_T^{-1} V_T x_T \subset U_T$. Then $x^{-1}[V_\tau, V_\sigma, V_T]x \subset [U_\tau, U_\sigma, U_T]$.

Hence the family $\mathcal{B}$ is a base of a semigroup topology on the group G. Denote this semigroup topology by ρ. The inclusion $\bigcap\{[U_\tau, U_\sigma, U_T] \cdot [U_\tau, U_\sigma, U_T]^{-1} : U_\tau \in \mathcal{B}_\tau, U_\sigma \in \mathcal{B}_\sigma, U_T \in \mathcal{B}_T\} \subset \{U_\sigma U_\sigma^{-1} \times U_T U_T^{-1} : U_\sigma \in \mathcal{B}_\sigma, U_T \in \mathcal{B}_T\} = \{(e_H, e_T)\}$ implies that the topology ρ is Hausdorff. Since the groups T and (H, σ) are saturated and the group T is nondiscrete, the group (G, ρ) is saturated too. According to [BR_1, Proposition 3] the base at the unit of the topology $\rho^\flat$ consists of the sets UU^{-1}, where $U \in \mathcal{B}$. Thus the topology $\rho^\flat$ coincides with the product topology of the groups $(H, \sigma) \times T^\flat$ and hence $(G, \rho^\flat) \in \mathcal{G}$ and H is a $\flat$-closed subgroup of the group G.

2. Proof of Theorem 2

The "if" part of Theorem 2 is trivial. To prove the "only if" part, suppose that T and (H, τ) are paratopological groups with the units e_T and e_H, satisfying the hypothesis of Theorem 2.

Using the Sorgenfrey property of the group T, choose an open invariant neighborhood U_0 of the unit e_T such that for any neighborhood $U \subset T$ of e_T there is a neighborhood $U' \subset T$ of e_T such that $x, y \in U$ for any elements $x, y \in U_0$ with $xy \in U'$. By induction we can build a sequence $\{U_n : n \in \omega\}$ of invariant open neighborhoods of e_T satisfying the following conditions:

(1) $\{U_n : n \in \omega\}$ is a neighborhood base at the unit e_T of the group T;

(2) $U_{n+1}^2 \subset U_n$ for every $n \in \omega$;

(3) for every $n \in \omega$ and any points $x, y \in U_0$ the inclusion $xy \in U_{n+1}$ implies $x, y \in U_n$;

(4) $\overline{U_n}^\flat \subsetneqq U_{n-1}$ for every $n \in \omega$, where $\overline{U_n}^\flat$ denotes the closure of the set U_n in the topology of $T^\flat$.

Remark that the condition (3) yields

(5) $(U_0 \backslash U_n) U_0 \cap U_{n+1} = \emptyset$ and hence $U_0 \backslash U_n \cap U_{n+1} U_0^{-1} = \emptyset$ for all n.

Since the group T is saturated, we can apply Proposition 3 of [BR_1] to conclude that the set $U_{n+2} U_0^{-1}$ is a neighborhood of the unit in $T^\flat$. Then the set $U_{n+2} U_{n+2} U_0^{-1} \subset U_{n+1} U_0^{-1}$ is a neighborhood of U_{n+2} in $T^\flat$. This observation together with (5) yields

(6) $\overline{U_0 \backslash U_n}^\flat \cap U_{n+2} = \emptyset$ for all n.

It follows from our assumptions on (H, τ) that there exists a group topology $\sigma \subset \tau$ on H such that the group (H, σ) belongs to the class $\mathcal{G}$ and (H, τ) has a neighborhood base $\mathcal{B}_\tau$ at the unit e_H consisting of sets, closed in the topology σ. By induction we can build a base $\{V_n : n \in \omega\}$ of open symmetric invariant neighborhoods of e_H in the topology σ such that $V_{n+1}^2 \subset V_n$ for every $n \in \omega$.

Consider the product $H \times T$ and identify H with the subgroup $H \times \{e_T\}$ of $H \times T$. It rests to define a topology on $H \times T$. At first we shall introduce an auxiliary sequence $\{W_k\}$ of "neighborhoods" of (e_H, e_T) satisfying the Pontryagin Conditions 1,2, and 4. For every $k \in \omega$ let

$$(\star) \quad W_n = \{(e_H, e_T)\} \cup \bigcup_{i > 2n} V_{ni} \times (U_{i-1} \setminus U_i)$$

and observe that $W_{n+1} \subset W_n$ for all n. Let us verify the Pontryagin Conditions 1,2,4 for the sequence (W_n).

To verify Conditions 1 and 2 it suffices to show that $W_n^2 \subset W_{n-1}$ for all $n \geq 1$. Fix any elements $(x,t),(x',t') \in W_n$. We have to show that $(xx',tt') \in W_{n-1}$. Without loss of generality, we can assume that $t,t' \neq e_T$. In this case we may find numbers $i,i' > 2n$ with $(x,t) \in V_{ni} \times (U_{i-1} \backslash U_i)$ and $(x',t') \in V_{ni'} \times (U_{i'-1} \backslash U_{i'})$. For $j = \min\{i,i'\}$ the Conditions (2), (5) imply

$$(xx',tt') \in V_{nj-1} \times (U_{j-2} \backslash U_{j+1}) \subset \bigcup_{k=j-1}^{j+1} V_{(n-1)k} \times (U_{k-1} \backslash U_k) \subset \bigcup_{k>2(n-1)} V_{(n-1)k} \times (U_{k-1} \backslash U_k) \subset W_{n-1}.$$

Taking into account that both the sequences $\{U_n\}$ and $\{V_n\}$ consist of invariant neighborhoods, we conclude that the sets W_n are invariant as well. Hence the Condition 4 holds too.

Now, using the sequence (W_n) we shall produce a sequence (O_n) satisfying all the Pontryagin Conditions 1–5. For every $n \in \omega$ put $O_n = \bigcup_{i=n}^{\infty} W_n W_{n+1} \cdots W_i$. Thus $W_n \supset O_{n+1} \supset W_{n+1}$ and $O_n \cap H \times \{e_T\} = \{(e_H, e_T)\}$ for all n. It is easy to see that the sequence $\{O_n\}$ consists of invariant sets and satisfies Pontryagin conditions 1–4. Hence the family $\{O_n\}$ is a neighborhood base at the unit of some (not necessarily Hausdorff) topology τ' on $G = H \times T$ turning G into a paratopological SIN-group. Applying Proposition 1.3 from [Ra$_1$] we conclude that the family $\mathcal{B}_\rho = \{OU : O \in \mathcal{B}_{\tau'}, U \in \mathcal{B}_\tau\}$ is a neighborhood base at the unit of some (not necessarily Hausdorff) semigroup topology ρ on G (here we identify H with the subgroup $H \times \{e_T\}$ in G). Since the topology ρ is stronger than the product topology π of the group $(H,\sigma) \times T^\flat$, the topology ρ is Hausdorff and H is a $\flat$-closed subgroup of the group (G,ρ). It follows from the construction of the topology ρ that $\rho|H = \tau$, $\chi(G,\rho) = \chi(H)$ and $|G/H| = |T|$.

At the end of the proof we show that the paratopological group (G,ρ) is saturated and $\flat$-regular. To show that the group (G,ρ) is saturated it suffices to find for every $n \geq 1$ nonempty open sets $V \subset (H,\sigma)$ and $U \subset T$ such that $V \times U^{-1} \subset W_n$. Taking into account that the group T is saturated and the set $U_{3n-1} \backslash \overline{U_{3n}}^\flat$ is nonempty, find a nonempty open set $U \subset T$ such that $U^{-1} \subset U_{3n-1} \backslash \overline{U_{3n}}^\flat$. Then $V_{3n^2}^{-1} \times U^{-1} \subset V_{3n^2} \times (U_{3n-1} \setminus U_{3n}) \subset W_n$. This implies that the group (G,ρ) is saturated and $(G,\rho^\flat) = (H,\sigma) \times T^\flat \in \mathcal{G}$.

The $\flat$-regularity of the group (G,ρ) will follow as soon as we prove that $\overline{W_n V}^\pi \subset W_{n-1} V$ for every $n \geq 2$ and $V \in \mathcal{B}_\tau$. Indeed, in this case, we shall get

$$\overline{O_{n+1} V}^\flat \subset \overline{O_{n+1} V}^\pi \subset \overline{W_n V}^\pi \subset W_{n-1} V \subset O_{n-1} V.$$

Fix any $x \in \overline{W_n V}^\pi$. If $x \in V \times \{e_T\}$, then $x \in W_{n-1} V$. Next, assume that $x \notin H \times \{e_T\}$. The property (4) of the sequence (U_k) implies that the point x has a π-neighborhood meeting only finitely many sets $H \times U_i$, $i \in \omega$. This observation together with $x \in \overline{W_n V}^\pi$ and $(\star)$ imply that $x \in \overline{V_{ni} V \times (U_{i-1} \setminus U_i)}^\flat$ for some $i > 2n$. The condition (6) implies

that the following chain of inclusions holds:

$$x \in \overline{V_{ni}V \times (U_{i-1} \setminus U_i)}^{\flat} \subset \overline{V_{ni}V}^{\sigma} \times \overline{(U_{i-1} \setminus U_i)}^{\flat} \subset V_{ni}^2 V \times (U_{i-2} \setminus U_{i+2}) \subset \bigcup_{j=i-1}^{i+2} V_{ni-1}V \times (U_{j-1} \setminus U_j) \subset \bigcup_{j>2n-2} V_{(n-1)j}V \times (U_{j-1} \setminus U_j) \subset W_{n-1}V.$$

Finally, assume that $x \in H \setminus V = (H \setminus V) \times \{e_T\}$. Since the set V is $\flat$-closed in H, there is $m \in \omega$ such that $V_m^{-1}V_m x \cap V = \emptyset$ and thus $V_m x \cap V_i V = \emptyset$ for all $i \geq m$. The inclusion $x \in \overline{W_n V}^{\pi}$ and $(\star)$ imply

$$(V_m \times U_m U_m^{-1})x \cap (V_{ni}V \times (U_{i-1} \setminus U_i)) \neq \emptyset$$

for some $i > 2n$. Then $V_m x \cap V_{ni}V \neq \emptyset$ and $U_m U_m^{-1} \cap (U_{i-1} \setminus U_i) \neq \emptyset$. In view of Property (5) of the sequence (U_k), the latter relation implies $m \leq i$. On the other hand, the former relation together with the choice of the number m yields $ni < m \leq i$ which is impossible. This contradiction finishes the proof of the inclusion $\overline{W_n V}^{\pi} \subset W_{n-1}V$.

3. Proof of Theorem 3

Given a topological space (X, τ) Stone [Sto] and Katetov [Kat] considered the topology τ_r on X generated by the base consisting of all canonically open sets of the space (X, τ). This topology is called the *regularization* of the topology τ. If (X, τ) is Hausdorff then (X, τ_r) is regular and if (X, τ) is a paratopological group then (X, τ_r) is a paratopological group too [Ra$_2$, Ex.1.9]. If (G, τ) is a paratopological group then τ_r is the strongest regular semigroup topology on the group G which is weaker than τ; moreover, for any neighborhood base $\mathcal{B}$ at the unit of the group (G, τ) the family $\mathcal{B}_r = \{\operatorname{int} \overline{U} : U \in \mathcal{B}\}$ is a base at the unit of the group (G, τ_r) [Ra$_3$, p.31–32]. The following proposition is quite easy and probably is known.

Proposition 6. *Let (X, τ) be a topological space. Then (X, τ) is pseudocompact if and only if the regularization (X, τ_r) is pseudocompact.*

For the proof of Theorem 3 we shall need a special pseudocompact functionally Hausdorff semigroup topology on the unit circle. We recall that a topological space X is *functionally Hausdorff* if continuous functions separate points of X.

Proposition 7. *There is a functionally Hausdorff pseudocompact first countable semigroup topology θ on the unit circle $\mathbb{T}$ which is not a group topology.*

Proof. Let $\mathbb{T}$ be the unit circle and $\chi : \mathbb{T} \to \mathbb{Q}$ be a (discontinuous) group homomorphism onto the groups of rational numbers. Fix any element $x_0 \in \mathbb{T}$ with $\chi(x_0) = 1$ and observe that $S = \{1\} \cup \{x \in \mathbb{T} : \chi(x) > 0\}$ is a subsemigroup of $\mathbb{T}$. Let θ be the weakest semigroup topology on $\mathbb{T}$ containing the standard compact topology τ and such that S is open in θ. It is easy to see that θ is functionally Hausdorff and the sets $S \cap U$, where $1 \in U \in \tau$, form a neighborhood base of the topology θ at the unit of $\mathbb{T}$.

By Proposition 6, to show that the group $(\mathbb{T}, \theta)$ is pseudocompact it suffices to verify that $\theta_r = \tau$. Since τ is a regular semigroup topology on the group $\mathbb{T}$ weaker than θ, we get $\theta_r \supset \tau$. To verify the inverse inclusion we first show that $\overline{U}^{\tau} = \overline{U}^{\theta}$ for any $U \in \theta$. Since $\tau \subset \theta$ it suffices to show that $\overline{U}^{\tau} \subset \overline{U}^{\theta}$. Fix any point $x \in \overline{U}^{\tau}$ and a neighborhood $V \in \tau$ of 1. We have to show that $x(V \cap S) \cap U \neq \emptyset$. Pick up any point $y \in xV \cap U$. Since U is open in the topology θ, we can find a neighborhood $W \in \tau$ of

1 such that $y(W \cap S) \subset xV \cap U$. Find a number N such that $\chi(yx_0^N) > \chi(x)$ and thus $yx_0^n \in xS$ for all $n \geq N$ (we recall that x_0 is an element of $\mathbb{T}$ with $\chi(x_0) = 1$). Moreover, since x_0 is non-periodic in $\mathbb{T}$, there exists a number $n \geq N$ such that $x_0^n \subset W$. Then $yx_0^n \in (yS \cap yW) \cap xS \subset (xV \cap U) \cap xS = x(V \cap S) \cap U$. Hence $x \in \overline{U}^\theta$ and $\overline{U}^\theta = \overline{U}^\tau$.

Then
$$\operatorname{int}_\theta \overline{U}^\theta = \mathbb{T}\backslash\overline{\mathbb{T}\backslash\overline{U}^\theta}^\theta = \mathbb{T}\backslash\overline{\mathbb{T}\backslash\overline{U}^\theta}^\tau \in \tau$$
which just yields $\theta_r \subset \tau$. □

Now we are able to present a *proof of Theorem 3*. The "if" part follows from the observation that for any Hausdorff pseudocompact paratopological group (G, τ) its group reflexion $G^\flat = (G, \tau_r)$ is a Hausdorff pseudocompact (and hence totally bounded) topological group [RR].

To prove the "only if" part, fix a Bohr-separated abelian paratopological group (H, τ) and let $\mathcal{B}_\tau$ be a neighborhood base at the unit of the group (H, τ). It follows that there is a group topology $\sigma' \subset \tau$ on H such that (H, σ') is totally bounded. Let $(\hat{H}, \sigma)$ be the Raikov completion of the group (H, σ'). It is clear that $\hat{H}$ is a compact abelian group and H is a normal dense subgroup of $\hat{H}$. It follows that $\mathcal{B}_\tau$ is a neighborhood base at the unit of some semigroup topology τ' on the group $\hat{H}$ with $\tau'|H = \tau$. Let $(\mathbb{T}, \theta)$ be the group from Proposition 7.

We shall define the topology on the product $G = \hat{H} \times \mathbb{T}$ as follows. Let $\mathcal{B}_\tau$, $\mathcal{B}_\sigma$ and $\mathcal{B}_\theta$ be the open neighborhood bases at the unit of the groups (H, τ), $(\hat{H}, \sigma)$ and $(\mathbb{T}, \theta)$ respectively. For arbitrary neighborhoods $U_\tau \in \mathcal{B}_\tau$, $U_\sigma \in \mathcal{B}_\sigma$ and $U_\theta \in \mathcal{B}_\theta$ with $U_\tau \subset U_\sigma$ let $[U_\tau, U_\sigma, U_\theta] = U_\tau \times \{e_\mathbb{T}\} \cup U_\sigma \times (U_\theta \backslash \{e_\mathbb{T}\})$, where e_H and $e_\mathbb{T}$ are the units of the groups H and $\mathbb{T}$ respectively. Denote by $\mathcal{B}$ the family of all such $[U_\tau, U_\sigma, U_\theta]$. Repeating the argument of Theorem 1 check that the family $\mathcal{B}$ is a base of some Hausdorff semigroup topology ρ on G. By π denote the topology of the product $(\hat{H}, \sigma) \times (\mathbb{T}, \theta_r)$. By Proposition 6 to show that the group (G, ρ) is pseudocompact it suffice to verify that $\rho_r \subset \pi$. For this we shall show that $\overline{U}^\rho \supset U_\sigma \times \overline{U_\theta}^\theta$ for every $U = [U_\tau, U_\sigma, U_\theta] \in \mathcal{B}$. Let $(x_{\hat{H}}, x_\mathbb{T}) \in U_\sigma \times \overline{U_\theta}^\theta$ and $V = [V_\tau, V_\sigma, V_\theta] \in \mathcal{B}$. It suffice to show that $\big((x_{\hat{H}}, x_\mathbb{T}) + V_\sigma \times (V_\theta \backslash \{e_\mathbb{T}\})\big) \cap U_\sigma \times (U_\theta \backslash \{e_\mathbb{T}\}) \neq \emptyset$. This intersection is nonempty if and only if the intersections $(x_{\hat{H}} + V_\sigma) \cap U_\sigma$ and $(x_\mathbb{T} + (V_\theta \backslash \{e_\mathbb{T}\})) \cap (U_\theta \backslash \{e_\mathbb{T}\})$ are nonempty. The first intersection is nonempty since $x_{\hat{H}} \in U_\sigma$ and the second is nonempty since $x_\mathbb{T} \in \overline{U_\theta}^\theta$ and the topology θ is non-discrete.

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